S.-T. Yau College Student Mathematics Contest

Applied Mathematics, Team, 2014

For the interval $[0,\pi]$, we divide it into $N+1$ equally spaced subintervals by using the nodal points:

$$0 = x_0 < x_1 < \cdots < x_{N+1} = \pi,$$

with

$$x_i = i\,h, \quad h = \pi/(N+1).$$

For any continuous function w on $[0,\pi]$, we define $\Pi_h w$ to be the piecewise linear interpolation of w, namely $\Pi_h w$ is linear on each subinterval (x_i, x_{i+1}) for $i = 0, 1, \cdots, N$, and it takes the same values as w at all nodal points x_i, $i = 0, 1, \cdots, N+1$. For any function w, we define

$$\|w\| = \Big(\int_0^\pi w^2(x)dx \Big)^{1/2}.$$

Prove the following estimates for any function $u \in C^2[0,\pi]$:

$$\|u - \Pi_h u\| \le \frac{1}{\pi^2} h^2 \|u''\|, \quad \|u' - (\Pi_h u)'\| \le \frac{1}{\pi} h \|u''\|.$$

claw-free graphs

A graph $G(V, E)$ is claw-free if it has no induced subgraph isomophic to the bipartite complete graph $K_{1,3}$,(i.e, $V = \{w, u_1, u_2, u_3\}$, $E = \{wu_1, wu_2, wu_3\}$).

Let G be a claw-free graph of order n. Let δ be the minimum degree of G and α the size of a maximum independent set. Prove that

$$\alpha \leq \frac{2n}{\delta + 2}.$$

Over $\Omega = (0,1)$, consider the heat equation with a homogeneous Dirichelt boundary condition

$$\partial_t u = u_{xx} + f, \quad \text{in } \Omega, \tag{1}$$
$$u(0,t) = u(1,t) = 0, \tag{2}$$

in which $f(x,t)$ is a given force term, with $\|f(\cdot,t)\|_{L^2} \leq M$, for any $t \geq 0$. The following semi-discrete implicit scheme is given

$$\frac{u^{n+1} - u^n}{\Delta t} = u_{xx}^{n+1} + f^{n+1}, \quad \text{in } \Omega, \tag{3}$$
$$u^{n+1}(0) = u^{n+1}(1) = 0, \tag{4}$$

in which u^k denotes the numerical solution at t^k, with $t^k = k\Delta t$, Δt being the time step size.

The final time is set as $T > 0$ and the initial data is given by $u^0(x)$. Prove the following uniform in time L^2 bound for the numerical scheme (3)-(4):

$$\left\|u^k\right\|_{L^2}^2 \leq \tilde{C} := \left\|u^0\right\|_{L^2}^2 + C_2^4 M^2, \quad \text{for any } k \geq 0, \tag{5}$$

in which $\tilde{C}$ is independent on the time step t^k, and C_2 is given by the following Pincaré inequality

$$\|v\|_{L^2} \leq C_2 \|v_x\|_{L^2}, \quad \text{if } v(0) = v(1) = 0. \tag{6}$$

Hint. Take an L^2 inner product with $2u^{n+1}$, use Poincaré inequality, and apply an induction in time to derive a uniform in time L^2 bound.